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Exact solution of a 1D quantum many-body system with momentum dependent interactions[1]

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Abstract

We discuss a 1D many-body model of *distinguishable* particles with local, momentum dependent two-body interactions. We show that the restriction of this model to fermions corresponds to the non-relativistic limit of the massive Thirring model. This fermion model can be solved exactly by a mapping to the 1D boson gas with inverse coupling constant. We provide evidence that this mapping is the non-relativistic limit of the duality between the massive Thirring model and the quantum sine-Gordon model. We also investigate the question if the generalization of this model to distinguishable particles is exactly solvable by the coordinate Bethe ansatz and find that this is not the case.

1. Introduction. In this paper we present, discuss, and solve a non-relativistic many-body system of particles moving in one space dimension (1D) and interacting with a particular local, momentum dependent two-body potential. As we will explain, this model is the natural fermion-analog of the 1D boson gas.

The 1D boson gas is one of the famous exactly solvable many-body models. It describes non-relativistic bosons moving in 1D and interacting with delta-function two-body interactions, and it was solved by Lieb and Liniger a long time ago [1] (a nice textbook discussion of this model and its solution can be found in Chapter I of reference [2]). We mention only in passing the considerable recent interest by experimental physicists triggered by a proposal of an experimental realization of this model in [4].

As mentioned, the particles in the 1D boson gas interact via a delta-function interaction. Due to the Pauli principle, this kind of interaction is trivial for fermions, and thus interesting

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fermion models with such an interaction require additional internal degrees of freedom [5, 6]. Our 1D many-body model is without internal degrees of freedom and with a particular local, translation invariant interaction which is non-trivial for fermions. It is defined by the following Hamiltonian ($\partial_{x_j} \equiv \partial/\partial x_j$),

$$H = -\sum_{j=1}^{N} \partial_{x_j}^2 + 2\lambda \sum_{j<k} \Big(\partial_{x_j} - \partial_{x_k}\Big) \delta(x_j - x_k) \Big(\partial_{x_j} - \partial_{x_k}\Big), \tag{1}$$

with an arbitrary number N of particles moving on the real line, $-\infty < x_j < \infty$ (we will also mention some generalizations of our results to an interval of length L with periodic or anti-periodic boundary conditions, $0 \leq x_j \leq L$); the real parameter λ determines the coupling strength. Note that the interactions depend not only on the particle distance $x_j - x_k$ but also the momentum difference $\hat{p}_j - \hat{p}_k \equiv -\mathrm{i}(\partial_{x_j} - \partial_{x_k})$. As we will see (Paragraph 2), due to this the Pauli principle is circumvented: this interaction is non-trivial on fermion wave functions, while it is trivial on boson wave functions. We will derive this model as non-relativistic limit of the massive Thirring model [7] (Paragraph 3), in the same way as the boson gas can be obtained as non-relativistic limit of ϕ^4-theory in 1+1 dimensions (see Appendix B.2). We find that this fermion model can be solved exactly by mapping it to the 1D boson gas with the coupling replaced by its inverse (Paragraph 4; as we will explain, this result is equivalent to the duality observed previously in [8]). This relation between our fermion model and the 1D boson gas is reminiscent to the famous duality between the massive Thirring model and the quantum sine-Gordon model [9], and we will present arguments that it actually is the non-relativistic limit of the latter (Paragraph 5). A natural question is if the generalization of this model to distinguishable particles remains exactly solvable by the coordinate Bethe Ansatz, but, as we shortly discuss in Paragraph 6, this is not the case.

Since the massive Thirring model is known to be integrable (in certain formal meanings of this word), it is perhaps not too surprising that its non-relativistic limit in equation (1) is exactly solvable. It thus is worth recalling that, despite of various interesting partial results [10, 11], the Thirring model has not been solved in full detail. It thus is interesting that its non-relativistic limit can be solved and studied by the much simpler methods which have been developed for the 1D boson gas.

In our derivations of non-relativistic limits in Paragraph 3 we start with the formal definition of the quantum massive Thirring model, perform expansions in $1/(\text{mass} \times c)$ with c the velocity of light, and we use *physical* arguments to justify our ignoring certain terms. In this way we arrive at a non-relativistic model which is well-defined, in the same spirit as reference [12]. It should be possible to make this procedure mathematically precise using the method proposed in [13]. As mentioned, the arguments in Paragraph 5 are somewhat heuristic. The other results are mathematically precise. We tried to keep the main text short, but for the convenience of the reader we included two appendices: In Appendix A we give a complimentary treatment of the singular interaction in our model, Appendix B contains details of our non-relativistic limits.

Remark: *In the first version of this paper (which unfortunately has already appeared in J. Phys. A: Math. Gen.) we presented an argument which seems to show that the generalized*

model defined by he Hamiltonian in equation (1) and for distinguishable particles is exactly solvable by the coordinate Bethe Ansatz [3]. This argument was incorrect.

The mistake leading to our wrong conclusions was a subtle point in Yang's arguments [3], and we were mislead by one of our sources to use an unfortunate notation covering up this point. To be more specific: while equation (C4) in the published version of this paper is not incorrect, it is misleading the way it is written, and our interpretation spelled out in Appendix C.1, Remark 2, led us astray: the twist we proposed to save the validity of the Yang-Baxter relations leads to an inconsistency at another point which we missed. The correct interpretation of this equation is now given in section 6.

It is worth mentioning that we discovered this mistake when we tried to generalize our result to other types of singular interactions, checking the validity of the Bethe Ansatz for the three particle case directly using symbolic computer programs written in MAPLE and MATHEMATICA (to be sure two of us wrote independent programs). While these programs nicely confirmed Yang's result for the delta interaction case, they showed that the coordinate Bethe Ansatz for our model is consistent if and only if the eigenfunction either has boson or fermion statistics.

2. Two particle case. To get a physical understanding of our model it is instructive to first consider the two-particle case $N = 2$. Introducing $x = x_1 - x_2$ and ignoring the trivial center-of-mass motion, H in equation (1) reduces to the following simple Hamiltonian,

$$h = -\partial_x^2 + 4\lambda \partial_x \delta(x) \partial_x, \tag{2}$$

whose eigenfunctions $\chi(x)$, $x \in \mathbb{R}$, are defined by satisfying $(\partial_x^2 + E)\chi(x) = 0$ for $x \neq 0$ and the following boundary conditions,

$$\begin{aligned} \chi'(0^+) - \chi'(-0^+) &= 0 \\ \chi(0^+) - \chi(-0^+) &= 4\lambda \chi'(0^+), \end{aligned} \tag{3}$$

with the prime indicating differentiation. Indeed, these are the boundary conditions obtained by integrating $h\chi = E\chi$ twice: first from $x = -0^+$ to $x > 0$ where $\chi'(0)$ is interpreted as the average of the left- and right derivative, and then once more from $x = -0^+$ to 0^+ (it is instructive to verify this formal argument by checking that the solutions below indeed satisfy $h\chi = E\chi$). The solutions of this are obtained by simple computations,

$$\begin{aligned} \chi_+(x) &= \cos(kx) \\ \chi_-(x) &= \frac{\sin(kx)}{2\lambda k} + \mathrm{sgn}(x)\cos(kx) \end{aligned} \tag{4}$$

with corresponding eigenvalue $E = k^2$. For real k these all are scattering states, and for $\lambda < 0$ there is one additional bound state for $k = \mathrm{i}/2\lambda$ with energy $E = -1/4\lambda^2$. Thus positive and negative values of λ correspond to the repulsive and attractive cases, respectively. As already mentioned, the boson wave function χ_+ is unchanged by the interaction, while the fermion wave function χ_- is modified, opposite to what happens for the delta-function interaction. It is worth noting that, in converting the interaction in equation (2) into the boundary conditions in equation (3), we have used a regularization

procedure which consistently avoids divergences which would occur in a naive treatment of this singular interaction (this is explained in more detail in Appendix A).

In a similar manner one finds that the eigenfunctions χ of the Hamiltonian in equation (1) for arbitrary N are given by the solutions of $(\sum_j \partial_{x_j}^2 + E)\chi(x_1,\ldots,x_N) = 0$ in all regions of non-coinciding points, together with the following boundary conditions

$$\begin{aligned} \Big(\partial_{x_j} - \partial_{x_k}\Big)\chi|_{x_j=x_k+0^+} &= \Big(\partial_{x_j} - \partial_{x_k}\Big)\chi|_{x_j=x_k-0^+} \\ \chi|_{x_j=x_k+0^+} - \chi|_{x_j=x_k-0^+} &= 2\lambda\Big(\partial_{x_j} - \partial_{x_k}\Big)\chi|_{x_j=x_k-0^+} \end{aligned} \tag{5}$$

(we used that $\partial_{x_j-x_k} = (\partial_{x_j} - \partial_{x_k})/2$). It is straightforward to check that these boundary conditions are trivially fulfilled for all non-interacting boson eigenfunctions $\chi_+ = \sum_{P\in S_N} \exp(\sum_j \mathrm{i}k_{Pj}x_j)$. They are, however, non-trivial for fermions.

3. Non-relativistic limit of the massive Thirring model. We now derive the non-relativistic limit of the massive Thirring model [7] and show that it is identical with the second quantization of the many-body Hamiltonians in equation (1). The Thirring model can be (formally) defined by the quantum field theory Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\mathrm{int}}$ where the free part is the usual Dirac Hamiltonian in 1D,

$$\mathcal{H}_0 = \int \mathrm{d}x \;:(\psi_+^\dagger, \psi_-^\dagger)\begin{pmatrix} -\mathrm{i}c\partial_x - E_0 & mc^2 \\ mc^2 & \mathrm{i}c\partial_x - E_0 \end{pmatrix}\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}: \tag{6}$$

with $m > 0$ the fermion mass, and the interaction is

$$\mathcal{H}_{\mathrm{int}} = 4g \int \mathrm{d}x \;: \psi_+^\dagger\psi_+\psi_-^\dagger\psi_- : \tag{7}$$

(see e.g. equation (2.1) in [10]) with g the coupling constant and the dots indicating normal ordering; the $\psi_\pm^{(\dagger)} \equiv \psi_\pm^{(\dagger)}(x)$ are fermion field operators obeying the usual canonical anticommutation relations (CAR) $\{\psi_\pm(x), \psi_\pm^\dagger(y)\} = \delta(x-y)$ etc., and E_0 is a parameter allowing us to change the reference energy which we will fix later to a convenient value. One can diagonalize $\mathcal{H}_0$ by Fourier transformation and diagonalization of a 2×2 matrix, which corresponds to a particular canonical transformation $(\psi_+^\dagger, \psi_-^\dagger) \to (\Psi_+^\dagger, \Psi_-^\dagger)$ (see Appendix B.1). We expand in powers of $1/mc$ and obtain, in position space,

$$\psi_\pm = \frac{1}{\sqrt{2}}\Big(\Psi_+ \pm \Psi_- \mp \frac{\mathrm{i}}{2mc}(\partial_x\Psi_+ \mp \partial_x\Psi_-) + \ldots\Big) \tag{8}$$

and $\mathcal{H}_0 = \mathcal{H}_0^+ + \mathcal{H}_0^-$ with $\mathcal{H}_0^\pm = \pm\int \mathrm{d}x \;: \Psi_\pm^\dagger[(mc^2 \mp E_0) - \partial_x^2/2m + \ldots]\Psi_\pm :$, where the dots are for higher order terms in $1/mc$. The positive- and negative states of the non-interacting model are now decoupled, and it is straightforward to compute the interaction in terms of the new fields $\Psi_\pm$. To obtain the non-relativistic limit we set $E_0 = mc^2$ and assume that mc^2 is large. In this case we can ignore the negative energy degrees of freedom Ψ_-: the non-interacting ground state is such that all the negative energy states are filled and the positive energy states empty (Dirac sea), and due to the large energy gap $2mc^2$ the interactions involving the filled states, in particular those across the gap, can be neglected if

one is only interested in the low-energy physics. We thus drop all terms in the Hamiltonian involving the fields $\Psi_-^{(\dagger)}$, and in leading non-trivial order in $1/mc$ we obtain the following Hamiltonian,

$$\mathcal{H}_{\rm non-rel} = \int {\rm d}x \, \frac{1}{2m}\Psi^\dagger(-\partial_x^2)\Psi + \frac{2g}{(2mc)^2} : \Big((\partial_x\Psi^\dagger)(\partial_x\Psi)\Psi^\dagger\Psi - \Psi^\dagger(\partial_x\Psi)(\partial_x\Psi^\dagger)\Psi \Big) : \quad (9)$$

with $\Psi \equiv \Psi_+$ obeying CAR and annihilating the non-interacting vacuum, $\Psi|0\rangle = 0$; we used $:[\Psi^\dagger(x)\Psi(x)]^2 := 0$, i.e., the lowest order term vanishes due to the Pauli principle, and thus the leading non-trivial interaction involves derivatives. It is straightforward to verify that this non-relativistic quantum field Hamiltonian $\mathcal{H}_{\rm non-rel}$ is the second quantization of our many-body Hamiltonian H in equation (1): for $2m = 1$ and $g/(2mc)^2 = -\lambda$, the eigenvalue equation $\mathcal{H}_{\rm non-rel}|N\rangle = E|N\rangle$ for N-particle states

$$|N\rangle = \int {\rm d}^N x \, \chi(x_1,\ldots,x_N)\Psi^\dagger(x_1)\cdots\Psi^\dagger(x_N)|0\rangle \quad (10)$$

is equivalent to $H\chi = E\chi$. Note that $\lambda < 0$ corresponds to $g > 0$, in agreement with what one should have expected from the fact that the massive Thirring model has bound states for $g > 0$ (see Eq. (2.15b) *ff* in [10]), whereas the sign of λ is such that the attractive case corresponds to $\lambda < 0$ (see Paragraph 2 above).

4. Solution I: Fermion model. We now determine all fermion eigenfunctions χ of the Hamiltonian in equation (1). Due to the fermion statistics we only need to determine $\chi = \chi(x_1, x_2, \ldots, x_N)$ in the fundamental wedge

$$\Delta_I : \quad x_1 < x_2 < \ldots < x_N. \quad (11)$$

For the same reason, the boundary conditions in the first line of Eq. (5) are automatically fulfilled, and the ones in the second line simplify to $2\chi|_{x_j=x_k+0^+} = 2\lambda(\partial_{x_j} - \partial_{x_k})\chi|_{x_j=x_k+0^+}$ where we only need to consider the cases $j = k+1$. Thus the equations determining our eigenfunctions are $(\sum_j \partial_{x_j}^2 + E)\chi = 0$ and

$$\Big(\partial_{x_{j+1}} - \partial_{x_j} - \frac{1}{\lambda}\Big)\chi|_{x_{j+1}=x_j+0^+} = 0. \quad (12)$$

Comparing with equations (2.1a), (2.4a) in [1] we see that *these conditions are identical with the ones determining the eigenfunctions of the 1D boson gas defined by the Hamiltonian*

$$H_B = -\sum_{j=1}^N \partial_{x_j}^2 + 2c_B \sum_{j<k} \delta(x_j - x_k) \quad (13)$$

at coupling

$$c_B = \frac{1}{\lambda} \quad (14)$$

in the fundamental wedge Δ_I. Since the latter eigenfunctions are well-known, we can immediately write down all eigenfunctions of our model

$$\chi(x_1, x_2, \ldots, x_N) = \prod_{1\le k<j\le N} \Big(\lambda[\partial_{x_j} - \partial_{x_k}] + 1\Big) \det_{1\le j,k\le N}[\exp({\rm i}k_j x_k)] \quad (15)$$

in Δ_I, and the corresponding eigenvalues are $E = \sum_j k_j^2$ (this explicit formula is apparently due to Gaudin [14]; see Chapter I in [2]).

In this paper we restrict ourselves to particles moving on the full line, but it is interesting to note that many of our results can be extended to the finite interval of length L, $0 \leq x_j \leq L$, with periodic or anti-periodic boundary conditions,

$$\chi(x_1, \ldots, x_N) = \mathrm{e}^{\mathrm{i}\eta}\chi(x_1 + L, \ldots, x_N), \tag{16}$$

and similarly for all other arguments x_j, with $\eta = 0$ or π. Similarly as for the 1D boson gas this yields the following conditions for the allowed momentum values,

$$\mathrm{e}^{\mathrm{i}k_j L} = (-1)^N \mathrm{e}^{\mathrm{i}\eta} \prod_{\ell=1}^{N} \frac{k_j - k_\ell + \mathrm{i}/\lambda}{k_j - k_\ell - \mathrm{i}/\lambda} \tag{17}$$

(these are the so-called Bethe equations; see e.g. Chapter I in [2]). Comparing with the Bethe equations for the 1D boson gas (equation (2.2) in [2]) we see that the duality above remains true for finite interval if we choose in our model periodic boundary condition ($\eta = 0$) if N is even and anti-periodic boundary conditions ($\eta = \pi$) if N is odd. In the thermodynamic limit $L, N \to \infty$ such that $\rho = N/L$ remains finite the difference in boundary conditions becomes irrelevant, and thus *all thermodynamic properties of our model are the same as the known thermodynamic properties of the 1D boson gas [16] at inverse coupling,* $c_B = 1/\lambda$. It would be interesting to know if there are any observables which can distinguish these two models.

5. Non-relativistic limit of the quantum sine-Gordon model. We now present evidence that the relation of our fermion model to the 1D boson gas found above is the non-relativistic limit of the duality between the massive Thirring model and the quantum sine-Gordon (qSG) model [9]. In the main text we will argue that the qSG model reduces to ϕ^4_{1+1}-theory for large (effective) mass. The result then follows since ϕ^4_{1+1}-theory in the non-relativistic limit is identical with the second quantization of the 1D boson gas (the details of this latter part of the argument are deferred to Appendix B.2).

The qSG model can be formally defined by the Hamiltonian $\mathcal{H}_{SG} = \mathcal{H}_0^B + \mathcal{H}_1^B$ with the usual free boson Hamiltonian

$$\mathcal{H}_0^B = \frac{1}{2}\int \mathrm{d}x \, : \left(c^2\Pi^2 + \phi[-\partial_x^2 + (mc)^2]\phi\right) : \tag{18}$$

and the interaction

$$\mathcal{H}_1^B = \int \mathrm{d}x \, : \frac{\alpha}{\beta^2}\left[1 - \cos\beta\phi\right] - \frac{(mc)^2}{2}\phi^2 : \tag{19}$$

with boson fields $\phi \equiv \phi(x) = \phi^\dagger$ and their conjugate variables $\Pi = \partial_t\phi/c^2$ obeying the usual canonical commutation relations (CCR), $[\Pi(x), \phi(y)] = -\mathrm{i}\delta(x-y)$ etc.; α and β are coupling parameters. It is important to note that, while the bosons in the qSG model are massless, the interaction generates a mass m with

$$(mc)^2 = \alpha. \tag{20}$$

We moved this mass term to the free part of the Hamiltonian so that the Taylor series of the interaction starts with the forth order term,

$$\mathcal{H}_1^B = \sum_{n=2}^{\infty} \frac{(-1)^{n-1}(mc)^2\beta^{2n-2}}{(2n)!} \int \mathrm{d}x \ :\phi^{2n}: \ . \tag{21}$$

In the non-relativistic limit we get, in leading order $1/mc$,

$$\phi = \frac{1}{\sqrt{2m}}\left(\Phi + \Phi^\dagger + \ldots\right) \tag{22}$$

where $\Phi^{(\dagger)}$ are non-relativistic boson fields obeying the CCR $[\Phi(x), \Phi^\dagger(y)] = \delta(x-y)$ (see Appendix B.2). Thus the coefficient in front of the n-th order term in the interaction is $\propto m^{2-n}\beta^{2n-2}c^2$, suggesting that, if the mass is large, one only needs to take into account the leading term $n = 2$ of the interaction. We thus conclude that, for large values of α, the qSG model has the same non-relativistic limit as ϕ^4_{1+1}-theory. Using that we find that the qSG Hamiltonian, in leading orders of $1/m$ and $1/c$, reduces to

$$\mathcal{H}^B_{\mathrm{non-rel}} = \int \mathrm{d}x \, \frac{1}{2m}\Phi^\dagger(x)(-\partial_x^2)\Phi(x) - \frac{(\beta c)^2}{16} :\Phi^\dagger(x)\Phi(x)\Phi^\dagger(x)\Phi(x): \tag{23}$$

where normal ordering is defined with respect to the vacuum $|0\rangle$ obeying $\Phi|0\rangle = 0$ (see Appendix B.2 for more details). This Hamiltonian now is well-defined, and for $2m = 1$ and $(\beta c/4)^2 = -c_B$ it is identical with the second quantization of the 1D boson Hamiltonian in equation (13) (see equations (1.1)–(1.12) in [2]).

In Paragraph 4 we found a duality between the fermion Hamiltonian defined in equation (1) and the boson Hamiltonian in equation (13) with the relation of coupling parameters given in equation (14). It is interesting to compare this to Coleman's duality between the massive Thirring model and the qSG model (see equation (1.9) in [9]),

$$\frac{4\pi}{\beta^2} = 1 + \frac{g}{\pi}. \tag{24}$$

Inserting the relations $\lambda = -g/c^2$ and $c_B = -(\beta c/4)^2$ which we obtained in the non-relativistic limits in Paragraph 3 and above, we obtain the relation in equation (14) up to a factor $\pi^2/4$ (the 1 on the r.h.s. in equation (24) disappears in the limit $c \to \infty$). We regard this agreement up to a numerical factor of order one as strong evidence that the the duality found in Paragraph 4 is indeed the non-relativistic limit of Coleman's duality (note that an exact agreement cannot be expected since we ignore the renormalization of parameters in the qSG and massive Thirring models [9]). Note, however, that this argument only applies to the *attractive* case $\lambda < 0$, whereas the duality in equation (14) is true also for $\lambda > 0$.

It is important to note that Coleman's duality provides also an identification of field operators in the qSG and the Thirring models (see equations (1.10) and (1.11) in [9]), but we do not see how this identification appears in our non-relativistic limits. We therefore regard the arguments in this paragraph only as a heuristic explanation of the duality in equation (14). It would be interesting to substantiate it in greater depth.

6. Generalization of the model to distinguishable particles. We now discuss the generalized model with the Hamiltonian in equation (1) but for *distinguishable* particles. We follow Yang [3] and make the following Bethe Ansatz for the eigenfunctions,

$$\chi = \sum_{P\in S_N} A_P(Q) \exp\left(\mathrm{i}\textstyle\sum_{j=1}^N k_{Pj}x_{Qj}\right) \quad \text{for } x_{Q1} < x_{Q2} < \ldots x_{QN}, \tag{25}$$

for all $Q \in S_N$, which implies $E = \sum_j k_j^2$. It is straightforward to adapt Yang's computation to our boundary conditions in equation (5). Inserting the Bethe Ansatz we obtain

$$\mathrm{i}(k_{Pi} - k_{P(i+1)})[A_{PT_i}(QT_i) - A_P(QT_i)] = \mathrm{i}(k_{Pi} - k_{P(i+1)})[A_P(Q) - A_{PT_i}(Q)] \tag{26}$$
$$A_P(QT_i) + A_{PT_i}(QT_i) - A_P(Q) - A_{PT_i}(Q) = 2\lambda\mathrm{i}(k_{Pi} - k_{P(i+1)})[A_P(Q) - A_{PT_i}(Q)] \tag{27}$$

where T_i is the transposition interchanging i and $i+1$. This implies

$$(1 - \mathrm{i}\lambda[k_{P(i+1)} - k_{Pi}])A_P(Q) = A_{PT_i}(QT_i) - \mathrm{i}\lambda[k_{P(i+1)} - k_{Pi}]A_{PT_i}(Q). \tag{28}$$

We now use that the permutation group S_N acts on the coefficients $A_P(Q)$ by the regular representation $R \to \hat{R}$ as follows,

$$A_P(QR) = (\hat{R}A_P)(Q) \equiv \sum_{Q'\in S_N} \hat{R}_{Q,Q'}A_P(Q') \tag{29}$$

with $\hat{R}_{Q,Q'} = \delta_{Q',QR}$. We thus can insert $A_{PT_i}(QT_i) = (\hat{T}_i A_{PT_i})(P)$ and write this latter relation as follows,

$$A_P = Y_i(k_{P(i+1)} - k_{Pi})A_{PT_i} \tag{30}$$

where A_P here is a vector with $N!$ elements $A_P(Q)$ and

$$Y_i(u) = \frac{\mathrm{i}u\hat{I} - (1/\lambda)\hat{T}_i}{\mathrm{i}u - 1/\lambda}. \tag{31}$$

As explained by Yang [3], consistency of the Bethe Ansatz is equivalent to $Y_i(u)$ satisfying certain consistency conditions. The first one (unitarity) is fulfilled in our case,

$$Y_i(-u)Y_i(u) = \hat{I}, \tag{32}$$

but the second one (Yang-Baxter relations) is not:

$$Y_i(v)Y_{i+1}(u+v)Y_i(u) \neq Y_{i+1}(u)Y_i(v+u)Y_{i+1}(v). \tag{33}$$

We conclude that the generalized model with distinguishable particles is not exactly solvable by the coordinate Bethe Ansatz.

7. Final comments. It is well-known that, in addition to the delta-function interaction which has been studied extensively in the context of integrable many-body systems, there are other local interactions which are physically very different [17]. Recently it was found that one particular such interaction leads to an exactly solvable many-body system of fermions in 1D which has a remarkable duality to the 1D boson gas [8]. In this paper we

found a natural physical interpretation of this fermion model: we showed that the boundary conditions used to define the model in [8] naturally arise from the N-body Hamiltonian in equation (1) which describes particles with local, momentum dependent two-body interactions. We also showed that this Hamiltonian arises as non-relativistic limit of the massive Thirring model, and we argued that the above-mentioned duality to the 1D boson gas comes from the well-known duality of the Thirring model to the quantum sine-Gordon model. We then proposed a generalization of this model where the particles are distinguishable, but we found that, different from the delta interactions case [3], this model cannot be solved exactly by the coordinate Bethe Ansatz.

As discussed in Chapter I.4 of [17], quantum mechanical point interactions in 1D leading to the boundary conditions in equation (3) have been studied extensively in the literature from a different point of view, and apparently it has been interpreted as a δ'-interaction (see [18]). The interpretation we give in this paper is very different and, as we hope to have convinced the reader, more natural.

We believe that our results show that, from a physical and mathematical point of view, the model defined in equation (1) is equally interesting as the delta-function interaction model given in equation (13). It thus would be worthwhile to explore this model further, e.g., extend our results to the finite interval with suitable boundary conditions etc.

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Appendix A: Physical interpretation of the interaction. In this Appendix we give a complimentary physical interpretation of the method to make sense of our momentum dependent interaction described in Paragraph 2. For simplicity we restrict ourselves to the 2-particle Hamiltonian h in equation (2).

In the main text we gave a formal argument converting the interaction in the Hamiltonian h to the boundary conditions in equation (3). It is interesting to note that, in doing this, we have specified a regularization procedure, i.e., given a consistent prescription avoiding divergences which would occur in a naive treatment of the singular interaction. Indeed, naively the action of h on a wave function $\chi(x)$ is $(h\chi)(x) = -\chi''(x) + 4\lambda\delta'(x)\chi'(0)$, but from our discussion in Paragraph 2 it is clear that h is also defined on wave function which are discontinuous at $x = 0$ and with $\chi'(0)$ therefore undefined. The above-mentioned regularization procedure amounts to replacing the ill-defined derivate at $x = 0$ by the well-defined average of the left- and right derivatives at $x = 0$, $\chi'(0) \to [\chi'(0^+) + \chi'(-0^+)]/2$. To see that this eliminates a divergence it is instructive to re-derive the bound state energy using Fourier transformation. The Fourier transform of $h\chi = E\chi$ can be written as

$$(k^2 - E)\hat{\chi}(k) = \lim_{\epsilon\to 0} 4\lambda k \int_{\mathbb{R}} \frac{\mathrm{d}q}{2\pi} \cos(\epsilon q) q \hat{\chi}(q) \tag{A1}$$

where the r.h.s. comes from the interaction with the factor $\cos(\epsilon q)$ providing the regular-

ization and the hat indicating Fourier transform. Computing from this $\hat{\chi}(k)$, multiplying with $k\cos(\epsilon k)$ and integrating we get the following self-consistency relation,

$$1 = 4\lambda \lim_{\epsilon\to 0} \int_{\mathbb{R}} \frac{\mathrm{d}q}{2\pi} \frac{\cos(\epsilon q) q^2}{q^2 + |E|} \tag{A2}$$

where we used that the bound state energy is negative, $E = -|E|$. Obviously, without the factor $\cos(\epsilon q)$ the integral on the r.h.s. is linearly divergent, but with this factor we obtain the well-defined result $1 = -2\lambda\sqrt{|E|}$, which for $\lambda = -|\lambda|$ has one solution. It is easy to see that this yields the same value for the bound state energy and the same bound state wave function which we obtained by a different method in Paragraph 2.

Appendix B. Non-relativistic limits: Details. In this Appendix we give more details about how to derive the non-relativistic limits of the Thirring model (Appendix B.1) and ϕ^4_{1+1}-theory (Appendix B.2) discussed in the main text.

B.1 Thirring model. The Dirac Hamiltonian in Eq. (6) in Fourier space is

$$\mathcal{H}_0 = \int \mathrm{d}k \ :(\hat{\psi}^\dagger_+, \hat{\psi}^\dagger_-) \begin{pmatrix} kc - E_0 & mc^2 \\ mc^2 & -kc - E_0 \end{pmatrix} \begin{pmatrix} \hat{\psi}_+ \\ \hat{\psi}_- \end{pmatrix} : \tag{B1}$$

with $\hat{\psi}^{(\dagger)}_\pm \equiv \hat{\psi}^{(\dagger)}_\pm(k)$ and the hat indicating Fourier transform. It is diagonalized with the following canonical transformation,

$$\hat{\psi}_\pm(k) = a_\pm(k)\hat{\Psi}_+(k) \pm a_\mp(k)\hat{\Psi}_-(k) \tag{B2}$$

where

$$a_\pm(k) = \sqrt{\frac{1}{2}\big(1 \pm \frac{kc}{E_k}\big)}, \quad E_k = \sqrt{(mc^2)^2 + (kc)^2}. \tag{B3}$$

This yields

$$\mathcal{H}_0 = \int \mathrm{d}k \ : \Big([E_k - E_0]\hat{\Psi}^\dagger_+(k)\hat{\Psi}_+(k) - [E_k + E_0]\hat{\Psi}^\dagger_-(k)\hat{\Psi}_-(k) \Big) : . \tag{B4}$$

Expanding this in powers of k/mc and transforming back to position space one obtains the equations given in (8) *ff* in the main text. Transforming the interaction in equation (7) to Fourier space, inserting the equations in (B2), and ignoring the terms involving the negative energy fields $\hat{\Psi}^{(\dagger)}_-$ we obtain

$$\mathcal{H}^+_{\mathrm{int}} = \frac{2g}{\pi} \int \mathrm{d}k_1 \cdots \mathrm{d}k_4 \, \delta(k_1 - k_2 + k_3 - k_4) v(k_1, \ldots, k_4) : \Psi^\dagger(k_1)\Psi(k_2)\Psi^\dagger(k_3)\Psi(k_4) : \tag{B5}$$

with the interaction vertex

$$\begin{aligned} v(k_1, \ldots, k_4) = \frac{1}{4}\big(& a_+(k_1)a_+(k_2)a_-(k_3)a_-(k_4) + a_+(k_3)a_+(k_4)a_-(k_1)a_-(k_2) \\ & - a_+(k_3)a_+(k_2)a_-(k_1)a_-(k_4) - a_+(k_1)a_+(k_4)a_-(k_3)a_-(k_2)\big) \end{aligned} \tag{B6}$$

which we (anti-) symmetrized using the CAR. Expanding this in powers of $1/mc$ we obtain

$$v(k_1,\ldots,k_4) = \frac{1}{(4mc)^2}(k_1-k_3)(k_2-k_4) + O((mc)^{-3}). \tag{B7}$$

Inserting this into equation (B5) and transforming back to position space we obtain the interaction term in the non-relativistic Hamiltonian given in equation (9).

B.2 ϕ^4_{1+1}-theory. This model can be formally defined by the Hamiltonian $\mathcal{H}^B = \mathcal{H}^B_0 + \mathcal{H}^B_{\rm int}$ with the free part $\mathcal{H}^B_0$ given in equation (18) in the main text and the interaction

$$\mathcal{H}^B_{\rm int} = g_B \int \mathrm{d}x \; :\phi^4: \tag{B8}$$

with boson fields ϕ and Π as defined after equation (18) in the main text, $m > 0$ the mass, and g_B the coupling; the dots indicate normal ordering to be specified below. The free boson Hamiltonian in equation (18) can be diagonalized in the usual manner,

$$\begin{aligned}\phi(x) &= c\int \frac{\mathrm{d}k}{\sqrt{2\pi}}\frac{1}{\sqrt{2E_k}}\left(\hat\Phi(k)\mathrm{e}^{\mathrm{i}kx} + \hat\Phi^\dagger(k)\mathrm{e}^{-\mathrm{i}kx}\right)\\ \Pi(x) &= -\frac{\mathrm{i}}{c}\int \frac{\mathrm{d}k}{\sqrt{2\pi}}\sqrt{\frac{E_k}{2}}\left(\hat\Phi(k)\mathrm{e}^{\mathrm{i}kx} - \hat\Phi^\dagger(k)\mathrm{e}^{-\mathrm{i}kx}\right)\end{aligned} \tag{B9}$$

with E_k as in equation (B3) and the $\hat\Phi^{(\dagger)}$ the Fourier transform of non-relativistic boson fields $\Phi^{(\dagger)}$ obeying the CCR $[\Phi(x),\Phi^\dagger(y)] = \delta(x-y)$ etc. This yields $\mathcal{H}^B_0 = \int \mathrm{d}k\, E_k \hat\Phi^\dagger(k)\hat\Phi(k)$ where, at this point, normal ordering is defined with respect to the non-interacting vacuum $|0\rangle$ obeying $\Phi(x)|0\rangle = 0$. Expanding in powers of $1/mc$ and transforming to position space we get

$$\mathcal{H}^B_0 = \int \mathrm{d}x\, \Phi^\dagger(x)[mc^2 - \partial_x^2/2m + O((mc)^{-1})]\Phi(x). \tag{B10}$$

To lowest non-trivial order in $1/mc$ the first equation in (B9) reduces to equation (22). Inserting this into the interaction in equation (B8) we get five terms, but only one of them commutes with the particle number operator $\hat N = \int \mathrm{d}x\, \Phi^\dagger(x)\Phi(x)$, namely $6g_B/(2m)^2 \int \mathrm{d}x\; :[\Phi^\dagger(x)\Phi(x)]^2:$. The other terms describe processes where the particle number is changed, and since the creation of particles requires an energy larger than mc^2 (according to equation (B10)) all these processes can be ignored in the non-relativistic limit where mc becomes large.[2] Thus the non-relativistic limit of ϕ^4_{1+1}-theory can be described the Hamiltonian

$$\mathcal{H}^B_{\rm non-rel} = \int \mathrm{d}x\, \frac{1}{2m}\Phi^\dagger(x)(-\partial_x^2)\Phi(x) + \frac{3g_B}{2m^2} :[\Phi^\dagger(x)\Phi(x)]^2: \tag{B11}$$

which, for $2m = 1$ and $3g_B/2m^2 = c_B$, is the second quantization of the 1D boson gas Hamiltonian given in equation (13).

[2]While this is physically plausible, we do not know a convincing mathematical argument to justify this simplification. We therefore regard this step as the weak link in our chain of arguments relating the SG model to the 1D boson gas.

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